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Equivariant connective K-theory for compact Lie groups

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Abstract

We construct a complex oriented, multiplicative, Noetherian, G-equivariant analogue of connective K-theory for an arbitrary compact Lie group G. Inverting the Bott element gives Atiyah–Segal equivariant K-theory and completion at the augmentation ideal gives non-equivariant connective K-theory of the Borel construction. We describe its role in the theory of equivariant formal group laws and calculate its coefficient ring for a variety of groups.

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1. Background

There are several ways of explaining the theoretical role of connective K-theory. It can be viewed as an embodiment of the universal ring for multiplicative formal group laws, as a bundle theory with triviality conditions [14], a continuous relative of algebraic K-theory, or as a refined home for certain indices. The usual construction is either by using an infinite loop space machine, or by using a brutal truncation of Atiyah's periodic complex K-theory.

In the equivariant world, these different roles suggest a variety of different constructions. The view of a bundle theory with a triviality condition, and the usual constructions of equivariant algebraic K-theory suggest the use of connective infinite loop space theory to give the brutal truncation $K[0,\infty)$ of Atiyah–Segal equivariant complex K-theory, which has coefficient ring $K[0,\infty)^G_* = R(G)[v]$, where $R(G)$ is the complex representation ring of G, and v is the Bott element of degree 2. For some purposes this

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is useful, but it has limitations. If G is of prime order, one may use a cell structure to calculate the cohomology of the G-space S^α (the one point compactification of a one dimensional faithful complex representation α); in particular we find

$$K[0,\infty)^G_0(S^\alpha) \cong \mathbb{Z} \neq 0 = K[0,\infty)^G_0(S^2).$$

This shows that the brutal truncation has the serious disadvantage that it is not complex orientable if $G \neq 1$. Similar calculations show that the brutal truncation does not satisfy the completion theorem either, whereas anyone calculating $ku^*(BG)$ (or reading [1]) will very quickly come to believe that $ku^*(BG)$ is the completion of a Noetherian $R(G)$-algebra. Finally, it appears from Carlsson's work on descent in algebraic K-theory that this type of construction is also inadequate for some purposes in algebraic K-theory.

In the present article we construct a G-equivariant analogue of connective K-theory for an arbitrary compact Lie group G. This is a complex oriented, ring valued theory from which Atiyah–Segal complex equivariant K-theory can be obtained by inverting the Bott class. Furthermore the ring $ku^*(BG)$ is the completion of the coefficient ring ku^*_G at the augmentation ideal, exactly as suggested by the calculations of [1]. In fact, this theory has all of the desirable properties listed in [9] except that its coefficients need not be in even degrees. Furthermore it agrees with the theory constructed there for the group of order p, and the construction here is simpler than that construction even in that special case. Apart from its value in understanding complex oriented cohomology theories and equivariant formal group laws, this theory can be used to define equivariant e-invariants with much smaller indeterminacy than periodic K-theory. The coefficient ring ku^*_G is closely related to the Rees ring (Section 2) of the representation ring for the augmentation ideal (and equal to it when G is a cyclic group). In another direction, the coefficient ring ku^*_G is closely related to the universal ring for multiplicative equivariant formal group laws [10] (and equal to it when G is a product of two cyclic groups).

In retrospect, this type of coefficient ring seems very natural. For our purposes, a G-spectrum X is a sequence of G-spaces $\underline{X}_V$ where V runs through finite dimensional complex representations V together with G-homeomorphisms $\Omega^V \underline{X}_{V\oplus W} \cong \underline{X}_W$; it is convenient to require all the representations to be sub-inner product spaces of a fixed 'complete G-universe' $\mathscr{U}$ (the sum of countably many copies of each simple complex representation). Since ku is complex orientable, we expect that ku has a representing G-spectrum with Vth term $\underline{ku}_V$ depending only on the dimension of a complex representation V. We then expect the first few terms to be analogous to the non-equivariant case, with

$$\underline{ku}_0 = \mathbb{Z} \times BU, \quad \underline{ku}_2 = BU, \quad \underline{ku}_4 = BSU,$$

where BU, BSU denote the classifying spaces for equivariant U or SU bundles (this is proved in Section 9). This gives calculations of some coefficient groups, since $ku^G_{-n} = \pi^G_0(\underline{ku}_n)$. Since bundles over a point are representations, we obtain

$$ku^G_0 = R(G), \quad ku^G_{-2} = J, \quad \text{and} \quad ku^G_{-4} = J_2,$$

where $J = \ker(R(G) \longrightarrow \mathbb{Z})$ consists of bundles over a point with virtual dimension 0, and J_2 consists of bundles over a point which are of virtual dimension 0 and of determinant 1. In particular this gives a geometric reason why ku is not equivariantly connective.

This work has its roots in [1], and the author thanks R.R. Bruner for many valuable conversations. G. Carlsson informs me that he has also considered a similar construction.

2. Statement of results

The main theorem is stated in terms of the representing G-spectrum ku for the cohomology theory, defined by the condition

$$\widetilde{ku}_G^*(X) = [X, ku]_G^*$$

for based G-spaces X. This notation follows the usual conventions for indicating actions and equivariant invariants for spaces. It is reasonable since, if we forget the group action, ku is equivalent to any other spectrum representing non-equivariant connective K-theory (see Theorem 2.1 (ii) below): up to homotopy *our ku means exactly the same as usual, but it has the additional structure of a group action*. Since equivariant homotopy groups are the non-equivariant homotopy groups of the Lewis–May fixed points, we have

$$ku_*^G = [S^0, ku]_*^G = \pi_*^G(ku) = \pi_*(ku^G)$$

and there is no danger of confusion. Similar remarks apply to periodic K-theory, and we write K for the representing G-spectrum for Atiyah–Segal equivariant K-theory as usual.

Theorem 2.1. *The G spectrum ku has the following properties.*

(i) *It is a strictly commutative ring spectrum (a commutative algebra over the equivariant sphere spectrum in the terminology of* [5])
(ii) *If H is any subgroup of G then if we view the G-equivariant spectrum ku as an H-spectrum we obtain the H-equivariant construction. In particular, ku is non-equivariantly the connective cover of the periodic K-theory spectrum K.*
(iii) *The ring G-spectrum ku is split, and hence* ku_G^* *is an algebra over* $ku^* = \mathbb{Z}[v]$.
(iv) *There is a ring map* $ku \longrightarrow K$ *of G-spectra which is localization to invert v.*
(v) *The coefficient ring* ku_G^* *is Noetherian. Its Krull dimension is equal to two or the maximal rank of an abelian subgroup of G, whichever is greater.*
(vi) *There is a ring map* $MU \longrightarrow ku$ *of G-spectra, so that ku is complex orientable (if G is abelian this agrees with the classical notion of orientability by* [4]).

The G-spectrum is constructed in Section 3. This construction makes Parts (i), (ii) and (iv) obvious. Part (iii) is proved in Section 4, Part (v) in Section 10 and Part (vi) in Section 11.

The connection with the non-equivariant theory is as good as possible in that the completion theorem and the local cohomology theorem hold. Since ku is a commutative algebra over the sphere spectrum with Noetherian coefficient ring, the usual proof given in [7,8] applies. Let $I = \ker(ku_G^* \longrightarrow ku^*)$ be the augmentation ideal.

Proposition 2.2. *For any compact Lie group G,*

(i) *the completion theorem holds for ku in the sense that*

$$ku^*(BG) = ku^*_G(EG) = (ku^*_G)^{\wedge}_I$$

(ii) *the local cohomology theorem holds for ku in the sense that there is a spectral sequence*

$$H^{*,*}_I(ku^G_*) \Rightarrow ku^G_*(EG) = \widetilde{ku}_*(BG^{ad}),$$

where the superscript ad denotes the Thom space of the adjoint bundle.

We have partial results about the coefficient ring. The most satisfactory of these are in terms of the ring homomorphism

$$p : ku^*_G \longrightarrow K^*_G = R(G)[v, v^{-1}]$$

comparing connective and periodic K-theory, where $R(G)$ is the complex representation ring of G. A standard construction in commutative algebra is the Rees ring Rees(R,J) associated to an ideal J of a ring R. This is the graded subring of the graded ring $R[v, v^{-1}]$ generated by R, v and $v^{-1} \cdot J$, where v has degree 2. It is thus R in degree 0 and each positive even degree, and it is J^n in degree $-2n$. We apply this when $R = R(G)$ and J is the augmentation ideal; in fact we need a slight modification. One may think of the Rees ring as a ring in which elements of J^n become divisible by v^n. In topology, the jth Chern class $c_j(V)$ of an n-dimensional complex representation V naturally lies in degree $-2j$, and $v^j c_j(V) = c^R_j(V)$, where

$$c^R_j(V) = \sum_{i=0}^{j} (-1)^i \binom{n-i}{n-j} \lambda^i(V) \in R(G)$$

is the representation theory Chern class. Thus we would expect the representation theory Chern class $c^R_j(V)$ of an n-dimensional representation to be divisible by v^j in ku^*_G, even if it is not in J^j. Accordingly we define an algebraic model for ku^*_G with this property.

Definition 2.3 (Bruner and Greenlees [1]). The modified Rees ring ModRees(G) is the subring of $R(G)[v, v^{-1}]$ generated by $R(G)$, v and $v^{-j} c^R_j(V)$ for all representations V.

Remark 2.4. (i) If G is abelian, then ModRees(G) = Rees$(R(G), J)$, but in general the inclusion

$$\mathrm{Rees}(R(G), J) \subseteq \mathrm{ModRees}(G)$$

is proper.

(ii) The Rees ring only depends on $R(G)$ as an augmented ring. However, the modified Rees ring also depends on the exterior powers, so we write it as a functor of G rather than the ring $R(G)$.

Example 2.5. The groups Q_8 and D_8 have isomorphic augmented representation rings, and hence also isomorphic Rees rings. Their modified Rees rings are not only different from the Rees ring, but also different from each other. Indeed, if V is the 2-dimensional simple representation of Q_8 then $c_2(V) = 2 - V$, whilst if W is the 2-dimensional simple representation of D_8 then $c_2(V) = 1 - V + r$ where r is the one dimensional representation with kernel the rotation subgroup.

We are now ready to state results about the coefficient rings: these are proved in Section 7. First we have a general statement if we invert v or tensor with the rational numbers.

Proposition 2.6. (i) *In positive degrees connective and periodic K-theory agree*:

$$ku_i^G = K_i^G \quad \text{if} \quad i \geqslant 0.$$

(ii) *Above degree* -6 *the coefficients are as follows*

$$ku_i^G = \begin{cases} 0 & \text{if } i \text{ is odd and } \geqslant 0, \\ R(G) & \text{if } i \text{ is even and } \geqslant 0, \\ 0 & \text{if } i = -1, \\ J & \text{if } i = -2, \\ 0 & \text{if } i = -3, \\ J_2 & \text{if } i = -4, \\ 0 & \text{if } i = -5, \end{cases}$$

where J is the augmentation ideal of representations of virtual dimension 0 *and J_2 is the subideal of representations of virtual dimension* 0 *and determinant* 1.

(iii) *Localized away from v connective and periodic K-theory agree*:

$$ku_G^*[1/v] \cong K_G^* = R(G)[v, v^{-1}].$$

(iv) *If G is finite, the map $ku_G^* \longrightarrow K_G^*$ is a rational monomorphism and the image is the rationalized modified Rees ring.*

Remark 2.7. The lower coefficients behave in a more complicated way: it follows from calculations for elementary abelian groups in [1], summarized in Section 8 that ku_{-6}^G can contain torsion (and is therefore not equal to the modified Rees ring in this degree) and ku_{-7}^G can be non-zero.

For groups of special forms we can give global descriptions.

Proposition 2.8. (i) *If G is topologically cyclic, ku_G^* is the Rees ring*

$$ku_G^* = \mathrm{Rees}(R(G), J).$$

(ii) *If G is a product of two topologically cyclic groups then ku_G^* is the representing ring for multiplicative equivariant formal group laws described in* [10]:

$$ku_G = L_G^m.$$

(iii) *If $G = U(n)$ then ku_G^* is the modified Rees ring*

$$ku_{U(n)}^* = \mathrm{ModRees}(U(n)).$$

Note that the calculation of $ku_{U(n)}^*$ allows us to define *ku*-theory Chern classes for arbitrary representations by naturality.

Corollary 2.9. *For any compact Lie group G, the image of the map*

$$ku_G^* \longrightarrow K_G^* = R(G)[v, v^{-1}]$$

contains ModRees(G).

Because of the central place of Chern classes, we prove Part (iii) in Section 5, and the rest of the proposition in Sections 6 and 7.

The calculations of [1] give calculations for a number of other groups.

The rest of the paper is devoted to proving the results we have just described. In Section 3 we describe the construction, and in Section 4 we show it has the desired multiplicative structure. Section 5 deals with $G = U(n)$ and hence establishes the existence of Chern classes, and Sections 6–10 deal with properties of the coefficient rings. Finally, in Section 11 we return to establish complex orientability.

3. The construction

The construction is inspired by the Hasse pullback square recovering the p-local integers from the p-adic integers and the rational numbers. The inputs to the construction are the usual Atiyah–Segal periodic K-theory G-spectrum K and the non-equivariant spectrum *ku*. We may inflate the non-equivariant spectrum *ku* to a G-spectrum $\inf_1^G ku$ by pullback along the quotient map $G \longrightarrow 1$ and change of universe to build in stability for non-trivial representations.

Definition 3.1. The equivariant connective K-theory spectrum *ku* is defined by the homotopy pullback square

$$\begin{array}{ccc} ku & \longrightarrow & K \\ \downarrow & & \downarrow \\ F(EG_+, \inf_1^G ku) & \longrightarrow & F(EG_+, K) \end{array}$$

of G-spectra.

The bottom horizontal is induced by the composite

$$\inf_1^G ku \longrightarrow \inf_1^G K \longrightarrow K,$$

where the first map is inverting v and the second arises since K is split. The right-hand vertical is induced by the unique map $EG \longrightarrow *$.

The rest of this article is devoted to proving the results of Section 2 by establishing properties of this spectrum. This should convince the reader that this is a good generalization of the non-equivariant spectrum.

Remark 3.2. (a) Part (ii) of Theorem 2.1 is immediate from the construction.

(b) If we use the Elmendorf–May highly structured inflation [6], then $\inf_1^G ku$ and $\inf_1^G K$ are commutative algebras over the sphere spectrum, and hence the spectra $F(EG_+, \inf_1^G ku)$ and $F(EG_+, \inf_1^G K) \simeq F(EG_+, K)$ are too. Since K is a commutative algebra over the sphere spectrum (for finite groups this follows by infinite loop space theory, and for general compact Lie groups it is proved in [12]) we may take the pullback in the category of commutative algebras over the sphere spectrum. This proves Part (i) of Theorem 2.1.

(c) Part (iv) of Theorem 2.1 follows from the fact that $ku^*(BG)[1/v] = K^*(BG)$ ([1, 1.1.1]) and hence that the bottom horizontal becomes an equivalence after inverting v.

(d) It is worth pausing to explain why this simple construction is plausible. Assuming ku exists, one expects there to be a homotopy pullback square

$$\begin{array}{ccc} ku & \longrightarrow & ku[1/v] \simeq K \\ \downarrow & & \downarrow \\ ku_v^\wedge & \longrightarrow & ku_v^\wedge[1/v] \end{array}$$

of G-spectra. Now $ku_v^\wedge \simeq \underleftarrow{\mathrm{holim}}_k\, ku/v^k$, and ku/v^k can be constructed from ku/v by a finite number of cofibre sequences. Finally this should represent an additive formal group law, for which the identity is the only point of finite order. Thus $ku/v = F(EG_+, ku/v)$, and hence

$$ku_v^\wedge \simeq F(EG_+, ku_v^\wedge) \simeq F(EG_+, ku).$$

(e) Analogous constructions can be made both for ko and for K-theory with reality in the sense of Atiyah, at least when the involution acts trivially on the group G. These will be considered elsewhere [2].

(f) Only very special (periodic) positively graded rings can be recovered from a localization simply by taking the part in positive degrees. For example Hopkins and Mahowald [11] construct the appropriate connective spectrum eo_2 from the Hopkins–Miller spectrum EO_2 by a pullback construction like the one we use here.

First we should record the fact that this is indeed a generalization of the one previously existing family of examples.

Lemma 3.3. *If G is of prime order, then the spectrum ku is equivalent to the one constructed in* [9].

Remark 3.4. The present construction is considerably simpler than the previous one. Nonetheless, I would never have recognized the present construction as the right one without the approach of [9].

Proof. Suppose G is of prime order. We continue to use ku to denote the form constructed here, and we let ku' denote the construction from [9]. Both constructions construct the equivariant form of ku as a homotopy pullback of a diagram of G-spectra. These diagrams do not appear to be directly comparable, but we have the diagram

$$\begin{tikzcd}
F(EG_+, \mathrm{inf}_1^G ku) \arrow[r] \arrow[d, "\simeq"] & F(EG_+, \mathrm{inf}_1^G K) \arrow[d] & K \arrow[l, "u"'] \arrow[d] \\
F(EG_+, \mathrm{inf}_1^G ku) \arrow[r] & F(EG_+, \mathrm{inf}_1^G K) \wedge \tilde{E}G & k \wedge \tilde{E}G \arrow[l, "u''"'] \\
F(EG_+, \mathrm{inf}_1^G ku) \arrow[r] \arrow[u, "\simeq"'] & F(EG_+, \mathrm{inf}_1^G ku) \wedge \tilde{E}G \arrow[u] & ku' \wedge \tilde{E}G \arrow[l, "u'"'] \arrow[u]
\end{tikzcd}$$

Here ku is the pullback of the top fork and ku' is the pullback of the bottom fork. We let ku'' denote the pullback of the middle fork. The left-hand verticals in the diagram are all equivalences, so we may concentrate on the two right-hand squares. We have comparison maps

$$\mathrm{fibre}(u) \longrightarrow \mathrm{fibre}(u'') \longleftarrow \mathrm{fibre}(u'),$$

and it suffices to show these are equivalences.

Since EG_+ is non-equivariantly S^0, the map u is a non-equivariant equivalence and hence the fibre of u is the fibre of u'', so that $ku \simeq ku''$. Now let K/ku' denote the cofibre of the map $ku' \longrightarrow K$. By the octahedral axiom, there is a cofibre sequence

$$F(\tilde{E}G, K/ku') \wedge \tilde{E}G \longrightarrow \mathrm{fibre}(u') \longrightarrow \mathrm{fibre}(u''),$$

so it remains to show $F(\tilde{E}G, K/ku') \wedge \tilde{E}G$ is contractible. However it is non-equivariantly contractible by construction so it suffices to shown its homotopy groups are zero.

First note that $\pi_i^G(K/ku') = 0$ for $i \geqslant 0$. Now $\tilde{E}G = \varinjlim_k S^{kV}$ where V is the reduced regular representation and

$$F(\tilde{E}G, K/ku') \wedge \tilde{E}G \simeq F(\tilde{E}G, K/ku') \simeq \operatorname{holim}_{\overleftarrow{k}} K/ku' \wedge S^{-kV}.$$

By complex orientability

$$K/ku' \wedge S^{-kV} \simeq \Sigma^{-2k} K/ku'$$

so that

$$\pi_s^G(K/ku' \wedge S^{-kV}) = 0 \text{ provided } s + 2k < 0,$$

and the Milnor exact sequence shows

$$\pi_s^G(F(\tilde{E}G, K/ku')) = 0$$

for each s as required. □

4. Multiplicative structure

In this section we show that ku is a split commutative ring spectrum up to homotopy. To show ku is a ring we need to describe a map $\mu : ku \wedge ku \longrightarrow ku$. The Mayer–Vietoris sequence of a homotopy pullback square gives a long exact sequence

$$\cdots \longrightarrow [EG_+ \wedge X, K]_1^G \longrightarrow [X, ku]_0^G \longrightarrow [X, K])_0^G \oplus$$

$$\left[EG_+ \wedge X, \inf_1^G ku\right]_0^G \longrightarrow [EG_+ \wedge X, K]_0^G \longrightarrow \cdots$$

for maps into ku. This is of practical use because $[EG_+ \wedge X, K]_1^G = 0$ in the cases we need to use.

Lemma 4.1. *If Z is a non-equivariant spectrum so that $H_*(Z)$ is a free $\mathbb{Z}$-module on even degree generators, then the graded group $[EG_+ \wedge Z, K]_*^G$ is zero in odd degrees.*

Proof. First note that

$$[EG_+ \wedge Z, K]_*^G = [BG_+ \wedge Z, K]_* = [Z, F(BG_+, K)]_*.$$

Now use the Atiyah–Hirzebruch spectral sequence together with the fact that $F(BG_+, K)$ has coefficients in even degrees. □

First we define a splitting map compatible with that of K.

Lemma 4.2. *There is a non-equivariant equivalence $i : \inf_1^G ku \longrightarrow ku$ so that*

$$\begin{array}{ccc} \inf_1^G ku & \xrightarrow{i} & ku \\ \downarrow & & \downarrow \\ \inf_1^G K & \xrightarrow{j} & K \end{array}$$

commutes, where j is the splitting for K.

Proof. Take $X = \inf_1^G ku$ in the Mayer–Vietoris sequence. To define i we must construct compatible maps $i' : \inf_1^G ku \longrightarrow K$ and $i'' : \inf_1^G ku \longrightarrow F(EG_+, \inf_1^G ku)$. For i' we use the composite

$$\inf_1^G ku \longrightarrow \inf_1^G K \xrightarrow{j} K,$$

and for i'' we use the map induced by projection $EG \longrightarrow *$. The choice of i' guarantees compatibility with j, and compatibility of (i', i'') follows. This constructs a non-equivariant equivalence i.

This specifies i uniquely by Lemma 4.1 since $H_*(ku)$ is a free $\mathbb{Z}$-module on even degree generators. □

Since we know that ku is a strictly commutative ring spectrum, we do not need to reprove that it has this property up to homotopy. However, it is convenient for Lemma 4.4 and Section 11 to have the details of the following proof to hand. This construction up to homotopy is visibly compatible with the highly structured version.

Lemma 4.3. *The G-spectrum ku is a commutative and associative ring spectrum up to homotopy.*

Proof. To show ku is a ring we must construct a map $\mu : ku \wedge ku \longrightarrow ku$. Taking $X = ku \wedge ku$ in the Mayer–Vietoris sequence, we see that we need to construct compatible maps $\mu' : ku \wedge ku \longrightarrow K$ and $\mu'' : ku \wedge ku \longrightarrow F(EG_+, \inf_1^G ku)$.

Since $ku[1/v] = K$, we may take μ' to be the composite

$$ku \wedge ku \quad \longrightarrow \quad K \wedge K \quad \longrightarrow \quad K,$$

where the last map is the multiplication for K. Since $ku \wedge ku \wedge EG_+ \simeq \inf_1^G(ku \wedge ku) \wedge EG_+$ we may take μ'' to correspond to the product in non-equivariant ku. These are compatible since the non-equivariant map $ku \longrightarrow K$ is a ring map, and K is split so that its equivariant and non-equivariant products are compatible. More precisely, noting that

$$\inf_1^G(A \wedge B) = \inf_1^G(A) \wedge \inf_1^G(B)$$

by [13, 3.14(iii)], compatibility follows because the diagram

$$\begin{tikzcd}
 & EG_+ \wedge ku \wedge ku \arrow[rr] & & EG_+ \wedge K \wedge K \arrow[dd] \\
EG_+ \wedge \inf_1^G (ku \wedge ku) \arrow[ur] \arrow[rr] \arrow[ddd] & & EG_+ \wedge \inf_1^G (K \wedge K) \arrow[ur] \arrow[ddd] & \\
 & & & EG_+ \wedge K \\
 & & & \\
EG_+ \wedge \inf_1^G ku \arrow[rr] & & EG_+ \wedge \inf_1^G K \arrow[uur] &
\end{tikzcd}$$

commutes. Applying Lemma 4.1 with $Z = ku \wedge ku$ we see that

$$[ku \wedge ku, F(EG_+, K)]_*^G = [ku \wedge ku \wedge BG_+, K]_*$$

is in even degrees, and hence the map $\mu : ku \wedge ku \longrightarrow ku$ corresponding to (μ', μ'') is unique. Since both μ' and μ'' are commutative it follows similarly that μ is commutative. Exactly analogous arguments with three factors ku in the domain show that μ is also associative. □

Finally we show the two bits of structure are compatible.

Lemma 4.4. *The splitting i is a ring map.*

Proof. We must show that the diagram

$$\begin{tikzcd} \inf_1^G ku \wedge \inf_1^G ku \arrow[r, "i \wedge i"] \arrow[d, "="'] & ku \wedge ku \arrow[dd, "\mu"] \\ \inf_1^G (ku \wedge ku) \arrow[d, "\inf_1^G (\mu)"'] & \\ \inf_1^G ku \arrow[r, "i"] & ku \end{tikzcd}$$

commutes. Once again, because $[\inf_1^G (ku \wedge ku) \wedge EG_+, K]^G_*$ is in even degrees by Lemma 4.1, it suffices to check after composing with $ku \longrightarrow K$ or $ku \longrightarrow F(EG_+, \inf_1^G ku)$ by Lemma 4.1 and this is clear. □

5. The unitary groups

The unitary groups have a universal role since all compact Lie groups may be embedded in them, and it is therefore valuable to understand the unitary case early in the analysis. In particular, once we have proved Proposition 2.8(iii):

$$ku^*_{U(n)} = \mathrm{ModRees}(U(n)),$$

we may define Chern classes by pullback from the unitary case.

Proof of 2.8(iii). The defining pullback square for ku gives a Mayer–Vietoris sequence

$$\cdots \longrightarrow ku^*_{U(n)} \longrightarrow K^*_{U(n)} \oplus ku^*(BU(n)) \xrightarrow{\langle i, -j\rangle} K^*(BU(n)) \longrightarrow \cdots.$$

For $U(n)$ we may identify all the groups and maps explicitly. Indeed,

$$R(U(n)) = \mathbb{Z}[c_1^R, c_2^R, \ldots, c_n^R, (\lambda^n)^{-1}],$$

and since ku and K are complex oriented, we have

$$K^*(BU(n)) = K^*[[c_1, c_2, \ldots, c_n]] \text{ and } ku^*(BU(n)) = ku^*[[c_1, c_2, \ldots, c_n]].$$

The critical fact is that the map $\langle i, -j\rangle$ is surjective. Indeed, an element of $K^{2n}(BU(n))$ may be written as a formal sum of terms $m_I/v^n \cdot (c^R)^I$ where $I = (i_1, i_2, \ldots)$ is a sequence of natural numbers almost all zero, m_I is an integer and

$$(c^R)^I = \prod_s (c_s^R)^{i_s} = \prod_s v^s c_s^{i_s}.$$

For all but finitely many terms the power $(\Sigma_s i_s) - n$ of v is positive. Subtracting off the finitely many negative terms using $\mathrm{im}(i)$ we are left with an element of $\mathrm{im}(j)$.

It now follows that $ku^*_{U(n)} = \ker(\langle i, -j\rangle)$, and since both i and j are monomorphisms

$$ku^*_{U(n)} = ku^*[[c_1, c_2, \ldots, c_n]] \cap K^*[c_1, c_2, \ldots, c_n, (\lambda^n)^{-1}],$$

which is the modified Rees ring as required. □

6. Filtrations on the representation ring

Consider the representation ring $R(G)$, and let J denote the augmentation ideal. We need to consider two filtrations: one associated to the Rees ring and the other to the modified Rees ring. The point of this section is to show that they define the same topology, so we can repeat the method used to prove Proposition 2.8(iii) in Section 5.

The first filtration is the J-adic filtration

$$R(G) \supseteq J \supseteq J^2 \supseteq J^3 \supseteq \cdots$$

by powers of the augmentation ideal. However for the modified Rees ring it is more relevant to use certain ideals generated by Chern classes. We therefore define a second filtration recursively by $J_0 = 0$ and take

$$J_n = J_1 J_{n-1} + J_2 J_{n-2} + \cdots + J_{n-1} J_1 + (c_n(V) \mid V \text{ is a representation})$$

to be the ideal generated by Chern classes of the appropriate degrees. Noting that $c_1(V) = |V| - V$, we see that $J = J_1$ is generated by first Chern classes. Again we have the filtration

$$R(G) \supseteq J = J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots.$$

Note that by definition

$$J_i J_j \subseteq J_{i+j}$$

so that this is a multiplicative filtration. The reason for considering these filtrations is that

$$\mathrm{Rees}(R(G))_{-2n} = J^n$$

and

$$\mathrm{ModRees}(G)_{-2n} = J_n.$$

Lemma 6.1. *The two topologies on $R(G)$ coincide.*

Proof. We have already noted that $J^n \subseteq J_n$, so that the J-adic topology is finer than the lower topology.

To see that the lower topology is finer than the J-adic topology note that, since $c_n(V)$ is non-equivariantly zero, it is clear that $J_i \subseteq J$. Now recall $c_\cdot(V \oplus W) = c_\cdot(V)c_\cdot(W)$. If the largest dimension of a simple representation is d it follows that $c_n(V)$ is decomposable whenever $n > d$, and therefore

$$J_{dk} \subseteq J^k. \qquad \square$$

Now consider the I-adic topology on ku_G^*, where $I = \ker(ku_G^* \longrightarrow ku^*)$ is the augmentation ideal. Since ku^* is zero in negative degrees, it follows that $I_{2n} = ku_{2n}^G$ for $n < 0$. On the other hand $ku_{2n}^G = K_{2n}^G$ for $n \geqslant 0$, so that $I_{2n} = J$ if $n \geqslant 0$.

Now let $\bar{I}(s)$ be the image of I^s in K_G^*. For each n this defines a topology on K_{2n}^G.

Lemma 6.2. *The $\bar{I}$ topology on K_{2n}^G coincides with the J-adic topology.*

Proof. First note that $\bar{I}(1)_{2n} = J$ if $n \geqslant 0$. Next, by Proposition 2.8(iii) (proved in Section 5), the Chern classes $c_n(V) \in K_{-2n}^G$ lift back to ku_*^G, and hence

$$J_n \subseteq \bar{I}(1)_{-2n} \subseteq J.$$

It follows that $\bar{I}(s+1)_{2n} = J^{s+1}$ if $n \geqslant 0$ and

$$J_n J^s \subseteq \bar{I}(s+1)_{-2n} \subseteq J^s.$$

The result follows from Lemma 6.1. $\square$

7. Coefficient rings

The pullback square defining ku automatically gives a Mayer–Vietoris sequence for its homotopy groups. However we show that in fact it induces a pullback square of rings and hence [1] gives a plentiful source of calculations: in most cases the calculations of $ku^*(BG)$ in [1] are stated so that the ring ku_G^* is given by omitting a visible completion. These will be recorded in Section 8; in the present section we apply the pullback theorem to prove the general statements about coefficient rings.

Theorem 7.1. *The defining pullback square induces a pullback square*

$$\begin{tikzcd} ku_G^* \arrow[r] \arrow[d] & K_G^* = R(G)[v, v^{-1}] \arrow[d] \\ ku^*(BG) \arrow[r] & K^*(BG) = R(G)_J^\wedge[v, v^{-1}] \end{tikzcd}$$

of commutative rings.

Proof. We must show that for each n the map

$$ku^{2n}(BG) \oplus R(G) \longrightarrow R(G)^{\wedge}_J$$

is surjective. Evidently it suffices to check that the image of $ku^{2n}(BG) \longrightarrow R(G)^{\wedge}_J$ contains $(J^s)^{\wedge}_J$ for some s. This follows from Lemma 6.2. □

Proof of 2.6. Most of Parts (i) and (ii) follow from the results about spaces in the representing spectrum in Section 9, but we give direct proofs here. In fact they follow from corresponding statements about $ku^n(BG)$. Indeed, $ku^n(BG) = K^n(BG)$ for $n \leqslant 0$, $ku^{-1}(BG) = ku^{-3}(BG) = 0$ and $ku^{-2}(BG) = J^{\wedge}_J$. To see this we let K/ku denote the cofibre of the map $ku \longrightarrow K$, and note that

$$(K/ku)^s(BG) = \begin{cases} 0 & \text{if } s \geqslant -1, \\ H^0(BG) & \text{if } s = -2, \\ H^1(BG) & \text{if } s = -3. \end{cases}$$

All this can be viewed as an argument with the Atiyah–Hirzebruch spectral sequence. To see that $ku^5(BG) = 0$ we need to argue that the differentials killing $K^5(BG)$ all originate in ku, or equivalently that none originate in $H^0(BG)$ or $H^2(BG)$. This is obvious for $H^0(BG)$, and for $H^2(BG)$ it follows from the fact that $H^2(BG)$ is generated by Chern classes. The statement about ku^G_{-4} follows by identifying the map $BU \longrightarrow BS^1$ as $Bdet$.

Part (iii) is immediate from Theorem 2.1(iv), which was proved in Section 3.

For Part (iv) it suffices to show that $(\widetilde{K/ku})^*(BG) \otimes \mathbb{Q} = 0$. However

$$(\widetilde{K/ku})^s(BG) = \lim_{\longleftarrow m} [BG^{(m)}, K/ku]^s = \lim_{\longleftarrow m} \lim_{\longrightarrow n} [BG^{(m)}, (K/ku)[-n, -2]]^s$$

with both direct and inverse limits being achieved at finite stages. Since $\tilde{H}^*(BG) \otimes \mathbb{Q} = 0$ and $(K/ku)[-n, -2]$ is a finite Postnikov tower, it follows that $(\widetilde{K/ku})^*(BG) \otimes \mathbb{Q} = 0$ as required. □

Proof of 2.8. Consider the relation between ku^*_G and the representing ring L^m_G for multiplicative equivariant formal group laws. First if $G = C$ is topologically cyclic, $ku^*(BC) \longrightarrow K^*(BC)$ is injective, so the defining pullback square identifies ku^*_C as a subring of K^*_C. From the formal group calculation of $ku^*(BC)$ we can directly see that the image of ku^*_C is the Rees ring, which was shown to be L^m_C in [10, 4.5(iii)].

Now suppose $A = C \times D$ where C and D are topologically cyclic. We claim

$$ku^*_A = L^m_A.$$

To prove this we use the fact that both rings can be constructed as pullbacks, and compare the forks. For ku^*_A the pullback property is established in Theorem 7.1. For L^m_A we use the Hasse square associated to the ideal (v); since L^m_A is Noetherian by [10], we have the pullback square

$$\begin{array}{ccc} L^m_A & \longrightarrow & L^m_A[1/v] = L^{sm}_A, \\ \Big\downarrow & & \Big\downarrow \\ (L^m_A)^\wedge_{(v)} & \longrightarrow & (L^m_A)^\wedge_{(v)}[1/v]. \end{array}$$

The equality $L^m_A[1/v] = K^m_A$ is [10, 4.5(ii)]; this ring is the universal ring L^{sm}_A of strictly multiplicative A-equivariant formal group laws. It now remains to show that $(L^m_A)^\wedge_{(v)} = ku^*(BA)$ and that the two forks are comparable. Both maps in the fork can be identified explicitly and seen to agree. Now by decoupling [10, 5.5] we have the tensor decomposition

$$L^m_A = L^m_C \otimes_{L^m} L^m_D,$$

which we may complete to obtain

$$(L^m_A)^\wedge_{(v)} = (L^m_C)^\wedge_{(v)} \hat{\otimes}_{L^m} (L^m_D)^\wedge_{(v)}.$$

On the other hand, because of the Gysin presentation for $ku^*(BC)$ when it is topologically cyclic, we have the Künneth theorem

$$ku^*(BA) = ku^*(BC) \hat{\otimes}_{ku_*} ku^*(BD).$$

By the completion theorem $ku^*(BG) = (ku^*_G)^\wedge_I$, so the result follows once observe that the I-adic and v-adic topologies on the Rees ring coincide.

Part (iii) was proved in Section 5. □

Remark 7.2. If $G=A$ is abelian we will see in Section 11 that ku is complex orientable, and that it has a canonical orientation, whose associated equivariant formal group law is multiplicative. It follows that there is a universal map $L^m_A \longrightarrow ku^*_A$, and it is this map that is an isomorphism when A is a product of two topologically cyclic groups.

8. Some specific examples

In this section we give the coefficient rings ku^G_* for a few groups not covered by the general theorems above. Using the pullback theorem 7.1, this is immediate from the calculations in [1].

Lemma 8.1. *If $ku^*(BG)$ has no v-torsion and is generated by Chern classes of representations, then*

$$ku^G_* = \mathrm{ModRees}(G).$$

Proof. By the pullback theorem, since the map $ku^*(BG) \longrightarrow K^*(BG)$ is injective, ku^*_G is just the intersection with $K^*_G = R(G)[v, v^{-1}]$. Since this intersection contains $\mathrm{ModRees}(G)$ by Corollary 2.9, the lemma follows. □

Certainly this applies to cyclic groups, but here are two more complicated examples.

Example 8.2. (i) (*Non-abelian pq*). If q is a prime and $p|q-1$ we let $G_{p,q}$ be the semidirect product with a normal subgroup C_q and quotient C_p acting by an automorphism of order p. By [1, 2.3.2] we have

$$ku_*^{G_{p,q}} = \mathrm{ModRees}(G_{p,q}).$$

(ii) (*Quaternion* 2-*groups*). For $n \geqslant 1$ we let $Q_{2^{n+2}}$ denote the quaternion group of order 2^{n+2}. By [1, 2.4.5] we have

$$ku_*^{Q_{2^{n+2}}} = \mathrm{ModRees}(Q_{2^{n+2}}).$$

Example 8.3. (*Dihedral* 2-*groups* [1, 2.5.4, 2.5.5]). We consider the dihedral group $D_{2^{n+2}}$ for $n \geqslant 1$. It is again shown in [1] that $ku^*(BG)$ is generated by Chern classes, so the image in periodic K-theory is the modified Rees ring, but now there is some v-torsion. In fact there is an extension

$$0 \longrightarrow T \longrightarrow ku_*^{D_{2^{n+2}}} \longrightarrow \mathrm{ModRees}(D_{2^{n+2}}) \longrightarrow 0$$

of modules over $\mathrm{ModRees}(D_{2^{n+2}})$, where $T = \Sigma^{-6}\mathbb{F}_2[a,b,d]/(ab+b^2)$ with a,b of degree -2 and d of degree -4. Since the dihedral group is a 2-group, the completion map is injective in periodic K-theory, and since the relations given in [1] are defined before completion they also give a presentation of ku_*^G, which we do not reproduce here.

Example 8.4. (*Elementary abelian* 2-*groups* [1, Chapter 4]). Let V denote an elementary abelian 2-group of rank r. Since it is a 2-group, completion is injective in periodic K-theory, and since the image of $ku^*(BV)$ in $K^*(BV)$ is generated by Chern classes the same is true before completion. Thus there is an exact sequence

$$0 \longrightarrow T \longrightarrow ku_V^* \longrightarrow \mathrm{ModRees}(V) \longrightarrow 0$$

of $\mathrm{ModRees}(V)$-modules. The module T is annihilated by 2 or by v, and its rather complicated structure is described in detail in [1]. For the present we only mention that T is zero if V is of rank 1, and more generally

$$T = T_2 \oplus T_3 \oplus \cdots \oplus T_r,$$

where T_i is non-zero in degree $-i-4$, and non-zero in all degrees $\leqslant -i-4$ with the same parity. In particular ku_{-6}^V contains torsion if $r \geqslant 2$ and $ku_{-7}^V \neq 0$ if $r \geqslant 3$.

Example 8.5. (*Alternating group* A_4 [1, 2.6.2]). Although $ku^*(BA_4)$ is not generated by Chern classes, its image in periodic K-theory is still generated by Chern classes. We therefore have a short exact sequence

$$0 \longrightarrow T \longrightarrow ku_*^{A_4} \longrightarrow \mathrm{ModRees}(A_4) \longrightarrow 0$$

of $\mathrm{ModRees}(A_4)$-modules, and T is non-zero in degree -6. A presentation may be read off from [1, 2.6.3]. Choose a non-trivial 1-dimensional representation α, and let τ

denote the 3-dimensional simple module. Now let $y = c_1(\alpha)$ and $v = c_3(\tau)$, $\mu = c_2(\tau)$, and let π be a new generator of degree -6. We then have the presentation

$$ku_*^{A_4} = ku_*[y, \mu, v, \pi]/I,$$

where

$$I = ([3](y), y\mu, yv, y\pi, 2v, vv, v\pi - 2\mu, v\mu^2 - 2\pi, \mu^3 - \pi^2 - \pi v - v^2).$$

Indeed, this follows from the v-adic Hasse square for the $R(G)$-module $ku_*[y, \mu, v, \pi]/I$ because one may check the v-adic and J-adic topologies coincide.

9. Some terms in the representing spectrum

We connect the spectrum ku to geometry by describing some of the terms in the representing Ω-spectrum. In particular this gives an alternative proof of Parts (i) and (ii) of (except for the statement that $ku^G_{-5} = 0$).

To be clear, for a unitary representation V we consider the group $U(V)$ of unitary maps of V and note that it admits an action of G by conjugation. Next, there is a map $U(V) \longrightarrow U(W)$ if there is an equivariant isometry $V \longrightarrow W$. Hence, using a complete complex universe $\mathscr{U}$ to organize the terms, we may form the direct limit

$$U = \varinjlim_{V \subseteq \mathscr{U}} U(V).$$

The result is compatible with restriction of the group of equivariance. Similarly we may form $SU(V)$ and SU.

Next, we may form the equivariant classifying spaces BU and BSU of equivariant unitary or special unitary bundles. There is a Grassmannian model in the unitary case as usual. Thus

$$BU_n = \mathrm{Gr}_n(\mathscr{U})$$

is the classifying G-space for n-dimensional unitary bundles and we may form $BU = \varinjlim_n BU_n$. As usual, we may construct BSU_n as the fibre of $Bdet : BU_n \longrightarrow BU_1$.

Lemma 9.1. *The following are terms in the Ω-spectrum for the G-spectrum ku*:

$$\underline{ku}_{-n} = \begin{cases} BU \times \mathbb{Z} & \text{if } n \text{ is even and } \geqslant 0, \\ U & \text{if } n \text{ is odd and } \geqslant -1, \\ BU & \text{if } n = -2, \\ SU & \text{if } n = -3, \\ BSU & \text{if } n = -4, \end{cases}$$

where BU, BSU are equivariant classifying spaces, and U, SU are groups of equivariant unitary or special unitary maps.

Proof. We may identify the terms $\underline{ku}_{-n}$ in an Ω-spectrum for ku by the same pullback square as defined ku itself. In fact, we claim that any map $X \longrightarrow Y$ with the property that for each subgroup H of G there is a complex H-representation V such that $\Omega^V X \longrightarrow \Omega^V Y$ is an equivalence, there is a pullback

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{map}(EG_+, X) & \longrightarrow & \mathrm{map}(EG_+, Y). \end{array}$$

The result will follow by taking $Y = \underline{K}_{-n}$, and X to be the candidate for $\underline{ku}_{-n}$. The point being that in each case this candidate is known to be correct non-equivariantly, so that the three terms of the fork are correct.

To prove the claim, note that the fibres of the verticals are

$$\mathrm{map}(\tilde{E}G, X) \longrightarrow \mathrm{map}(\tilde{E}G, Y).$$

Now we argue as usual: by induction on the group order we may assume that this is an H equivariant equivalence for all proper subgroups. Now take

$$S^{\infty W} = \varinjlim_{W^G=0} S^W$$

and note that it is H-contractible for all proper subgroups H, but has G-fixed point set S^0. Now consider the map

$$\tilde{E}G \longrightarrow S^{\infty W}$$

which is an equivalence in G fixed points. Its mapping cone may be constructed using cells G/H_+ for proper subgroups H, so it suffices to show

$$\mathrm{map}(S^{\infty W}, X) \longrightarrow \mathrm{map}(S^{\infty W}, Y)$$

is an equivalence, which follows from the hypothesis. □

10. The Noetherian condition

In this section we prove Part (v) of Theorem 2.1, stating that ku^*_G is Noetherian. The statement on Krull dimensions then follows from [1, Chapter 1]. We use the short exact sequence

$$0 \longrightarrow T \longrightarrow ku^*_G \longrightarrow Q \longrightarrow 0$$

of ku^*_G-modules, where T is the v-power torsion and $Q = \mathrm{im}(ku^*_G \longrightarrow K^*_G)$.

In fact we choose a faithful representation of G in $U(n)$ and show that ku^*_G is finitely generated as a module over the Noetherian ring $\mathrm{Ch} = ku^*_{U(n)} = R(U(n))[v, c_1, \ldots, c_n]$. It suffices to show that T and Q are finitely generated Ch-modules.

We let $\hat{\mathrm{Ch}} = ku^*(BU(n)) = R(U(n))[v][[c_1, \ldots, c_n]]$ denote the completion of Ch.

Proposition 10.1. *The v-power torsion module T is a finitely generated* Ch*-module.*

Proof. From the defining pullback square, T is also equal to the v-power torsion in $ku^*(BG)$. Since $ku^*(BG)$ is a finitely generated $\hat{\mathrm{Ch}}$-module by Quillen's argument, it follows that its submodule T is also finitely generated.

Since the chain of annihilators of $v, v^2, v^3, \ldots$ is an ascending chain of $\hat{\mathrm{Ch}}$-modules, it follows that $v^k T = 0$ for some k. Now if $t_1, \ldots, t_N$ is a generating set for T as an $\hat{\mathrm{Ch}}$-module we claim that it is also a generating set as a Ch-module. Indeed, if $t = \sum_i f_i t_i$ with $f_i \in \hat{\mathrm{Ch}}$ we may write

$$f_i = \sum_I a_{i,I} c^I$$

with $a_{i,I} \in R(U(n))[v]$ where the sum is over finite sequences $I = (i_1, i_2, \ldots)$ with $i_s \geqslant 0$ and $c^I = \prod_s c_s^{i_s}$. Since $|c^I| \longrightarrow -\infty$ as $|I| \longrightarrow \infty$ it follows that $|a_{i,I}| \longrightarrow \infty$ as $|I| \longrightarrow \infty$ and the power of v dividing $a_{i,I}$ is greater than k for all but finitely many terms. Hence the power series $f_i \in \hat{\mathrm{Ch}}$ can be replaced by polynomials $f_i' \in \mathrm{Ch}$ and still have

$$t = \sum_i f_i' t_i$$

as required. □

Proposition 10.2. *The module Q is a finitely generated* Ch*-module.*

Proof. From the defining pullback square, Q may be calculated by

$$Q = \hat{Q} \cap R(G)[v, v^{-1}],$$

where $\cap R(G)[v, v^{-1}]$ indicates the inverse image under the natural map $R(G) \longrightarrow R(G)^{\wedge}_J$. Each Q_{2n} is a submodule of the Noetherian ring $R(G)$ and hence finitely generated. The central point is the fact that $\hat{Q}$ is rationally generated in the sense of the following lemma.

Lemma 10.3.

$$\hat{Q}_{2n} = (Q_{2n})^{\wedge}_J.$$

We pause to note that this is not a triviality: for example it is easy to find a non-zero sub-$\mathbb{Z}_p^{\wedge}$-module $T \cong \mathbb{Z}_p^{\wedge}$ of $\mathbb{Z}_p^{\wedge} \oplus \mathbb{Z}_p^{\wedge}$ with $T \cap (\mathbb{Z} \oplus \mathbb{Z}) = 0$.

Proof. It suffices to prove the result completed at p for each integer prime p, and we work completed at p for the rest of the proof.

Suppose first that G is finite. Note that $\hat{Q}_{-2n}$ is a sub-$\mathbb{Z}_p^{\wedge}$-module of $\hat{J}$, and that it is necessarily free. The point is that $\hat{Q}_{-2n}$ and $\hat{J}$ have equal ranks and $\hat{J}$ is the completion of J. Now if we choose a $\mathbb{Z}_p^{\wedge}$-basis of $\hat{Q}_{-2n}$ and express it in terms of the basis of J, we may put it into diagonal form. Multiplying by p-adic units, the diagonal entries may be taken to be powers of p. These integral basis elements lie in Q_{-2n} and their completion gives $\hat{Q}_{-2n}$ as required.

If G is not finite we argue similarly, except that J must be replaced by a direct summand of finite codimension to ensure equality of ranks. The summand is the kernel of the composite

$$K_G^{2n} \longrightarrow K^{2n}(BG) \longrightarrow (K/ku)^{2n}(BG),$$

where we note that $(K/ku)^{2n}(BG)$ is a finitely generated abelian group. □

Returning to the proof of Proposition 10.1, we note that since $\hat{Q}_{\geqslant 0} = (R(G))^{\wedge}_J[v]$ and $\hat{Q}_{-2} = J^{\wedge}_J$ the corresponding statements, $Q_{\geqslant 0} = R(G)[v]$ and $Q_{-2} = J$ follow for Q. Accordingly $Q_{\geqslant -2n}$ generates a finitely generated ring for each $n \geqslant 0$, and it remains to show that this is the whole of Q for n sufficiently large.

For each n we may consider the image $\hat{P}_{-2n} \subseteq \hat{Q}_{-2n}$ of products of negative degree elements. We may now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_{-2n} \arrow[r] \arrow[d] & Q_{-2n} \arrow[r] \arrow[d] & \bar{Q}_{-2n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{P}_{-2n} \arrow[r] & \hat{Q}_{-2n} \arrow[r] & \hat{\bar{Q}}_{-2n} \arrow[r] & 0
\end{tikzcd}$$

Since J-completion preserves sums and is exact on finitely generated modules, it follows that $\hat{P}_{-2n}$ is the completion of P_{-2n}, and hence that $\hat{\bar{Q}}_{-2n}$ is the completion of $\bar{Q}_{-2n}$.

Next note that $J\bar{Q}_{-2n} = 0$ since P_{-2n} contains

$$Q_{-2} \cdot Q_{-2n+2} = J \cdot Q_{-2n+2} \supseteq J \cdot Q_{-2n}.$$

Hence $\bar{Q}_{-2n}$ is J-complete and

$$\bar{Q}_{-2n} = \hat{\bar{Q}}_{-2n}.$$

Since $\hat{Q}$ is finitely generated it follows that $\hat{Q}_{-2n} = \hat{\bar{Q}}_{-2n} = 0$ for sufficiently large n as required. □

11. Complex orientability

We must construct a ring map $v : MU \longrightarrow ku$. The Mayer–Vietoris sequence of a homotopy pullback square gives a long exact sequence

$$\cdots \longrightarrow [EG_+ \wedge X, K]^G_1 \longrightarrow [X, ku]^G_0 \longrightarrow$$

$$[X, K]^G_0 \oplus [EG_+ \wedge X, \inf_1^G ku]^G_0 \longrightarrow [EG_+ \wedge X, K]^G_0 \longrightarrow \cdots.$$

Accordingly, to construct v, we need to construct compatible maps $v' : MU \longrightarrow K$ and $v'' : MU \longrightarrow F(EG_+, \inf_1^G ku)$.

For v'' we use the composite

$$MU \longrightarrow F(EG_+, MU) \simeq F(EG_+, \inf_1^G MU) \longrightarrow F(EG_+, \inf_1^G ku),$$

where the first is induced by $EG \longrightarrow *$, the second is the equivalence induced by the fact that MU is split, and the third is induced by the non-equivariant map $MU \longrightarrow ku$. These may all be chosen to be maps of algebras over the sphere spectrum.

For v' we prove a lemma.

Lemma 11.1. *There is a ring G-map $MU \longrightarrow K$ which is non-equivariantly the Todd genus.*

We will turn to the proof when we have described how this implies the theorem.

The compatibility condition for the maps v' and v'' is described by their images in $[MU \wedge EG_+, K]_G$. By the following result it is practical to check this, since taking $Z = MU^{\wedge i}$ in Lemma 4.1 we obtain.

Lemma 11.2. *The graded group $[EG_+ \wedge MU^{\wedge i}, K]_*^G$ is zero in odd degrees.*

The compatibility condition is therefore given by the rectangle

$$\begin{tikzcd} MU \arrow[r] \arrow[d] & F(EG_+, \inf_1^G MU) \arrow[r] & F(EG_+, \inf_1^G ku) \arrow[d] \\ K \arrow[rr] & & F(EG_+, \inf_1^G K) \end{tikzcd}$$

or equivalently, by the rectangle

$$\begin{tikzcd} MU \wedge EG_+ \arrow[r] \arrow[d] & MU \arrow[r] & ku \arrow[d] \\ K \wedge EG_+ \arrow[rr] & & K. \end{tikzcd}$$

Embedding this in the diagram

$$\begin{tikzcd} \inf_1^G MU \wedge EG_+ \arrow[rr] \arrow[dr, "\simeq"] \arrow[dd] & & \inf_1^G MU \arrow[dr] \arrow[dd] & \\ & MU \wedge EG_+ \arrow[rr, crossing over] & & MU \arrow[dd] \\ \inf_1^G ku \wedge EG_+ \arrow[rr] \arrow[dr, "\simeq"] & & \inf_1^G ku \arrow[dr] & \\ & ku \wedge EG_+ \arrow[rr] \arrow[from=uu, crossing over] & & ku \arrow[d] \\ & & & K \end{tikzcd}$$

we see that commutativity follows from the non-equivariant description of $MU \longrightarrow ku$ and the compatibility of splittings for the pair (ku, K) for the pair (MU, K).

Uniqueness of the lift v of (v', v'') follows since $[EG_+ \wedge MU, K]_1^G = 0$. Similarly the fact that v is a commutative and associative ring map follows from commutativity and associativity after smashing with EG_+ or inverting v together with the fact that $[EG_+ \wedge MU^{\wedge i}, K]_1^G = 0$ for $i = 2, 3$. □

Proof of 11.1. From the Atiyah–Segal completion theorem

$$K_G^0(BU(n)) = R(G \times U(n))^{\wedge}_{J(G,U(n))}$$

where $J(G, U(n))$ is the ideal in $R(G \times U(n))$ of elements restricting to zero on every subgroup $H \subseteq G \times U(n)$ intersecting $1 \times U(n)$ trivially. In fact

$$J(G, U(n)) = (e^R(V \otimes \lambda^1), e^R(V \otimes \lambda^2), \ldots, e^R(V \otimes \lambda^n) \,|\, V \text{ is a representation of } G),$$

where λ^i is the ith exterior power of the natural representation of $U(n)$.

If V is an n-dimensional representation of G we have

$$MU(V) = \mathrm{Gr}_n(V \oplus \mathscr{U})^{\gamma_n}$$

where $\mathscr{U}$ is a complete complex universe and γ_n is the natural n-plane bundle. If we identify $\mathrm{Gr}_n(V \oplus \mathscr{U})$ with $BU(n)$, the K-theory Thom class of γ_n is $e^R(V \otimes \lambda^1)$, so

$$K_G^*(MU(V)) = e^R(V \otimes \lambda^1) \cdot (R(G \times U(n))^{\wedge}_{J(G,U(n))}.$$

Because of the multiplicative property, these Thom classes are compatible under suspension as V varies and hence give rise to an element of

$$K_G^0(MU) = \varprojlim_V K_G^0(MU(V)),$$

where the absence of a $\varprojlim^1$-term is due to the fact that $K_G^1(MU(V)) = 0$.

The fact that these give a ring map again follows from multiplicativity of Euler classes. The non-equivariant behaviour follows since the Euler classes are compatible under forgetting equivariance. □

Remark 11.3. It seems likely that v can be chosen to be a map of commutative algebras over the sphere spectrum, but there are some questions of compatibility to be decided. To start with, the given map v'' can be chosen to be a highly structured ring map. Next, Joachim [12, Appendix A] constructs a highly structured Atiyah–Bott–Shapiro map $MSpin^c \longrightarrow K$, and there is a model for MU giving a highly structured ring map $MU \longrightarrow MSpin^c$. It remains to check that the two implied structures on $F(EG_+, \inf_1^G K)$ agree.

Finally we should discuss the equivariant formal group law [3] associated to ku when $G = A$ is abelian. It is conventional to use the G-space $\mathbb{C}P(\mathscr{U})$ of complex lines in the complete A-universe $\mathscr{U}$ as a model for the classifying space BU_1 for A-line bundles.

Lemma 11.4. *The natural map*

$$ku_A^*(\mathbb{C}P(\mathscr{U})) \longrightarrow K_A^*(\mathbb{C}P(\mathscr{U}))$$

is injective in cohomological degree 2, *and the canonical complex orientation* $y(\varepsilon) := e^R(z)/v$ *lies in the image.*

Proof. The fact that $e^R(z)/v$ is in the image is immediate by considering the diagram

$$\begin{tikzcd} ku^*_{A\times U(1)} \arrow[r] \arrow[d] & K^*_{A\times U(1)} \arrow[d] \\ ku^*_A(\mathbb{C}P(\mathscr{U})) \arrow[r] & K^*_A(\mathbb{C}P(\mathscr{U})) \end{tikzcd}$$

The injectivity follows from Proposition 2.6(ii) and the Artin–Rees lemma. □

Lemma 11.5. *The element* $y(\varepsilon) \in ku^2_A(\mathbb{C}P(\mathscr{U}))$ *is a complex orientation and the associated universal group law is multiplicative.*

Proof. In the range where we need to calculate $ku^*_A(\mathbb{C}P(\mathscr{U}))$ and $ku^*_A(\mathbb{C}P(\mathscr{U})\times\mathbb{C}P(\mathscr{U}))$ are embedded in the corresponding groups for periodic K-theory; for $ku^*_A(\mathbb{C}P(\mathscr{U}))$ this is in Lemma 11.4, and for $ku^*_A(\mathbb{C}P(\mathscr{U}) \times \mathbb{C}P(\mathscr{U}))$ we have the analagous diagram

$$\begin{tikzcd} ku^*_{A\times U(1)\times U(1)} \arrow[r] \arrow[d] & K^*_{A\times U(1)\times U(1)} \arrow[d] \\ ku^*_A(\mathbb{C}P(\mathscr{U}) \times \mathbb{C}P(\mathscr{U})) \arrow[r] & K^*_A(\mathbb{C}P(\mathscr{U}) \times \mathbb{C}P(\mathscr{U})) \end{tikzcd}$$ □

References

[1] R.R. Bruner, J.P.C. Greenlees, Connective K-theory of finite groups, Mem. Amer. Math. Soc., 165 (2003) no. 785, 127pp.

[2] R.R. Bruner, J.P.C. Greenlees, Connective real K-theory of finite groups, in preparation.

[3] M.M.Cole, J.P.C. Greenlees, I. Kriz, Equivariant formal group laws, Proc. London Math. Soc. 81 (2000) 355–386.

[4] M.M. Cole, J.P.C. Greenlees, I. Kriz, The topological universality of equivariant bordism, Math. Z., 239 (2002) 455–475.

[5] A.D. Elmendorf, I. Kriz, M. Mandell, J.P. May, Rings Modules and Algebras in Stable Homotopy, AMS, Providence, RI, USA, 1997.

[6] A.D. Elmendorf, J.P. May, Algebras over equivariant sphere spectra, J. Pure Appl. Algebra 116 (1997) 139–149.

[7] J.P.C. Greenlees, K-homology of universal spaces and local cohomology of the representation ring, Topology 32 (1993) 295–308.

[8] J.P.C. Greenlees, Augmentation ideals of equivariant cohomology rings, Topology 37 (1998) 1313–1323.

[9] J.P.C. Greenlees, Equivariant forms of connective K-theory, Topology 38 (1999) 1075–1092.

[10] J.P.C. Greenlees, Multiplicative equivariant formal group laws, J. Pure Appl. Algebra 165 (2001) 183–200.

[11] M.J. Hopkins, M.E. Mahowald, From elliptic curves to homotopy theory, preprint, 1998.
[12] M. Joachim, Higher coherences for equivariant K-theory, preprint, 2001, 27pp.
[13] L.G. Lewis, J.P. May, M. Steinberger (with contributions by J.E. McClure), Equivariant Stable Homotopy Theory, Lecture Notes in Mathematics, Vol. 1213, Springer, New York, 1986.
[14] G.B. Segal, K-homology Theory and Algebraic K-Theory, Lecture Notes in Mathematics, Vol. 575, Springer, New York, 1977, pp. 113–127.